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The Egorov theorem for transverse Dirac-type operators on foliated manifolds☆

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Abstract

Egorov's theorem for transversally elliptic operators, acting on sections of a vector bundle over a compact foliated manifold, is proved. This theorem relates the quantum evolution of transverse pseudodifferential operators determined by a first-order transversally elliptic operator with the (classical) evolution of its symbols determined by the parallel transport along the orbits of the associated transverse bicharacteristic flow. For a particular case of a transverse Dirac operator, the transverse bicharacteristic flow is shown to be given by the transverse geodesic flow and the parallel transport by the parallel transport determined by the transverse Levi-Civita connection. These results allow us to describe the noncommutative geodesic flow in noncommutative geometry of Riemannian foliations.

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0. Introduction

The Egorov theorem is a fundamental fact in microlocal analysis and quantum mechanics. It relates the evolution of pseudodifferential operators on a compact manifold (quantum observables) determined by a first-order elliptic operator with the corresponding evolution of classical observables — the bicharacteristic flow on the space of symbols. More precisely, let M be a compact manifold and let P be a positive, self-adjoint, elliptic, first-order pseudodifferential operator on M with the positive principal symbol $p \in S^1(T^*M \setminus 0)$. Let f_t be the bicharacteristic flow of the operator P, that is, the Hamiltonian flow of p on T^*M. Egorov's theorem [8] states that, for any pseudodifferential operator A of order 0 with the principal symbol $a \in S^0(T^*M \setminus 0)$, the operator $A(t) = \mathrm{e}^{\mathrm{i}tP} A \mathrm{e}^{-\mathrm{i}tP}$ is a pseudodifferential operator of order 0. The principal symbol $a_t \in S^0(T^*M \setminus 0)$ of this operator is given by the formula

$$a_t(x,\xi) = a(f_t(x,\xi)), \quad (x,\xi) \in T^*M \setminus 0.$$

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In the particular case $P = \sqrt{\Delta_g}$, where Δ_g is the Laplace–Beltrami operator of a Riemannian metric g on M, the corresponding bicharacteristic flow is the geodesic flow of g on T^*M.

In [16], Egorov's theorem was extended to pseudodifferential operators acting on sections of a vector bundle E on a compact manifold M. First, the authors gave an invariant definition of the subprincipal symbol of a positive, self-adjoint, elliptic, first-order pseudodifferential operator $P \in \Psi^1(M, E)$ with the real scalar principal symbol $p \in S^1(T^*M \setminus 0)$ as a partial connection along the Hamiltonian vector field of p on T^*M. The parallel transport along the orbits of the Hamiltonian flow of p on T^*M defined by this partial connection determines a flow β_t acting on $S^0(T^*M \setminus 0, \mathrm{End}\,(\pi^*E))$. Then the theorem in [16] says that for any operator $A \in \Psi^0(M, E)$ with the principal symbol $a \in S^0(T^*M \setminus 0, \mathrm{End}\,(\pi^*E))$, the operator $A(t) = \mathrm{e}^{\mathrm{i}tP} A \mathrm{e}^{-\mathrm{i}tP}$ is in $\Psi^0(M, E)$, and its principal symbol $a_t \in S^0(T^*M \setminus 0, \mathrm{End}\,(\pi^*E))$ is given by $a_t = \beta_t(a)$.

If $P = \sqrt{\Delta_g}$, where Δ_g is the Hodge–Laplace operator of a Riemannian metric g acting on differential forms on M, the corresponding flow β_t on $S^0(T^*M \setminus 0, \pi^*\mathrm{End}\,(\Lambda^*_{\mathbb{C}} T^*M))$ is given by the parallel transport along the orbits of the geodesic flow of g on T^*M with respect to the Levi-Civita connection.

We also mention the works [1,4,9,10,27] (and references therein) for discussion of Egorov's theorem for matrix-valued operators and relations to parallel transport.

On the other hand, in [23] the author proved a version of Egorov's theorem for scalar transversally elliptic operators on compact foliated manifolds. For this purpose, I used the transverse pseudodifferential calculus developed in [21]. The associated algebra of symbols is a noncommutative, Connes-type operator algebra associated with a natural foliation $\mathcal{F}_N$ on the conormal bundle $N^*\mathcal{F}$ of the foliation $\mathcal{F}$. The Egorov theorem stated in [23] relates the quantum evolution of transverse pseudodifferential operators determined by a first-order transversally elliptic operator P with the (classical) evolution of its symbols determined by the transverse bicharacteristic flow of P, which is the restriction of the bicharacteristic flow of P to $N^*\mathcal{F}$. We also mention related works on the Duistermaat–Guillemin trace formula: [22] for transversally elliptic operators on Riemannian foliations and [28] for the basic Laplacian of a Riemannian foliation.

The main purpose of this paper is to extend Egorov's theorem to transversally elliptic operators acting on sections of a holonomy equivariant vector bundle on a compact foliated manifold, using ideas of [16]. In this case, it is shown that the corresponding classical evolution is given by the parallel transport along the orbits of the transverse bicharacteristic flow. Furthermore, we introduce a natural class of first-order transversally elliptic operators, namely, transverse Dirac operators $D_{\mathcal{E}}$ with coefficients in an arbitrary holonomy equivariant Hermitian vector bundle $\mathcal{E}$, and compute the transverse bicharacteristic flow for these operators. Quite remarkably, the associated parallel transport is naturally determined by the transverse Levi-Civita connection.

The transverse Dirac operators were introduced in [11]. These papers mainly concern with the transverse Dirac operators acting on basic sections (see also [12,13,17–19] and references therein). The index theory of transverse Dirac operators was studied in [5]. Finally, spectral triples defined by transverse Dirac-type operators on Riemannian foliations were studied in [21,23]. In particular, the results of this paper can be considered as a complement of our study of the noncommutative geodesic flow started in [23].

1. Classes of transverse pseudodifferential operators

Throughout in the paper, $(M, \mathcal{F})$ is a compact foliated manifold, E is a Hermitian vector bundle on M, $\dim M = n$, $\dim \mathcal{F} = p$, $p + q = n$.

We will consider pseudodifferential operators, acting on half-densities. For any vector bundle V on M, denote by $|V|^{1/2}$ the associated half-density vector bundle. Let $C^\infty(M, E)$ denote the space of smooth sections of the vector bundle $E \otimes |TM|^{1/2}$, $L^2(M, E)$ the Hilbert space of square integrable sections of $E \otimes |TM|^{1/2}$, $\mathcal{D}'(M, E)$ the space of distributional sections of $E \otimes |TM|^{1/2}$, $\mathcal{D}'(M, E) = C^\infty(M, E)'$, and $H^s(M, E)$ the Sobolev space of order s of sections of $E \otimes |TM|^{1/2}$. Finally, let $\Psi^m(M, E)$ denote the standard classes of pseudodifferential operators, acting in $C^\infty(M, E)$.

We will use the classes $\Psi^{m,-\infty}(M, \mathcal{F}, E)$ of transversal pseudodifferential operators. Let us briefly recall the definition, referring the reader to [21] for more details.

First, for any $k_A \in S^m(I^p \times I^p \times I^q \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^r))$ (here $r = \mathrm{rank}\, E$), define an operator $A : C^\infty_c(I^n, \mathbb{C}^r) \to C^\infty(I^n, \mathbb{C}^r)$ by the formula

$$Au(x, y) = (2\pi)^{-q} \int e^{i(y-y')\eta} k_A(x, x', y, \eta) u(x', y')\, dx'\, dy'\, d\eta, \tag{1}$$

where $u \in C_c^\infty(I^n, \mathbb{C}^r)$, $x \in I^p$, $y \in I^q$. The function k_A is called the complete symbol of A. As usual, we will consider only classical (or polyhomogeneous) symbols, that is, those symbols, which can be represented as an asymptotic sum of homogeneous (in η) components.

Let $\varkappa : U \subset M \to I^p \times I^q$, $\varkappa' : U' \subset M \to I^p \times I^q$ be a pair of compatible foliated charts on M equipped with trivializations of the bundle E over them. Any operator A of the form (1) with the Schwartz kernel, compactly supported in $I^n \times I^n$, determines an operator $A : C_c^\infty(U, E|_U) \to C_c^\infty(U', E|_{U'})$, which extends to an operator in $C^\infty(M, E)$ in a trivial way. The resulting operator is called an elementary operator of class $\Psi^{m,-\infty}(M, \mathcal{F}, E)$.

Definition 1.1. The class $\Psi^{m,-\infty}(M, \mathcal{F}, E)$ consists of all operators A in $C^\infty(M, E)$ which can be represented in the form

$$A = \sum_i A_i + K,$$

where A_i are elementary operators of class $\Psi^{m,-\infty}(M, \mathcal{F}, E)$, corresponding to a pair $\varkappa_i$, $\varkappa_i'$ of compatible foliated charts, $K \in \Psi^{-\infty}(M, E)$.

2. The principal symbol for transverse ΨDOs

In this section, we will recall the definition of the principal symbol for an operators of class $\Psi^{m,-\infty}(M, \mathcal{F}, E)$.

First, we define the principal symbol of A given by (1) as the leafwise half-density

$$\sigma(A)(x, x', y, \eta) = k_{A,m}(x, x', y, \eta) |dx|^{1/2} |dx'|^{1/2}, \tag{2}$$

where $k_{A,m}$ is the degree m homogeneous component of the complete symbol k_A.

Before giving the global definition of the principal symbol, we recall several notions (for more details, see e.g. [24] and references therein). Let $\gamma : [0.1] \to M$ be a continuous leafwise path in M with the initial point $x = \gamma(0)$ and the final point $y = \gamma(1)$ and T_0 and T_1 arbitrary smooth submanifolds (possibly, with boundary), transversal to the foliation, such that $x \in T_0$ and $y \in T_1$. Sliding along the leaves of the foliation $\mathcal{F}$ determines a diffeomorphism $H_{T_0 T_1}(\gamma)$ of a neighborhood of x in T_0 to a neighborhood of y in T_1, called the holonomy map along γ. The differential of $H_{T_0 T_1}(\gamma)$ at x gives rise to a well-defined linear map $T_x M / T_x \mathcal{F} \to T_y M / T_y \mathcal{F}$, which is independent of the choice of transversals T_0 and T_1. This map is called the linear holonomy map and denoted by $dh_\gamma : T_x M / T_x \mathcal{F} \to T_y M / T_y \mathcal{F}$. The adjoint of dh_γ yields a linear map $dh_\gamma^* : N^*\mathcal{F}_y \to N^*\mathcal{F}_x$, where we denote by $N^*\mathcal{F}$ the conormal bundle to $\mathcal{F}$.

Denote by G the holonomy groupoid of $\mathcal{F}$. Recall that G consists of $\sim_h$-equivalence classes of continuous leafwise paths in M, where we set $\gamma_1 \sim_h \gamma_2$ if γ_1 and γ_2 have the same initial and final points and the same holonomy maps. G is equipped with the source map $s : G \to M$, $s(\gamma) = \gamma(0)$, and the range map $r : G \to M$, $r(\gamma) = \gamma(1)$.

Let $\mathcal{F}_N$ be the linearized foliation in $\tilde{N}^*\mathcal{F} = N^*\mathcal{F} \setminus 0$ (cf., for instance, [25]). The leaf of the foliation $\mathcal{F}_N$ through $\nu \in \tilde{N}^*\mathcal{F}$ is the set of all points $dh_\gamma^*(\nu) \in \tilde{N}^*\mathcal{F}$ where $\gamma \in G$, $r(\gamma) = \pi(\nu)$ (here $\pi : T^*M \to M$ is the bundle map).

Consider a foliated chart $\varkappa : U \subset M \to I^p \times I^q$ on M with coordinates $(x, y) \in I^p \times I^q$ (I is the open interval $(0, 1)$) such that the restriction of $\mathcal{F}$ to U is given by the sets $y = \text{const}$, equipped with a trivialization of the vector bundle E. We will always assume that the foliated chart $\varkappa$ is regular, which means that it admits an extension to a foliated chart $\bar{\varkappa} : V \subset M \xrightarrow{\sim} (-2, 2)^n$ with $\bar{U} \subset V$. There is the corresponding chart in T^*M with coordinates written as $(x, y, \xi, \eta) \in I^p \times I^q \times \mathbb{R}^p \times \mathbb{R}^q$. In these coordinates, the restriction of the conormal bundle $N^*\mathcal{F}$ to U is given by the equation $\xi = 0$. So we have a chart $\varkappa_n : U_1 \subset N^*\mathcal{F} \xrightarrow{\sim} I^p \times I^q \times \mathbb{R}^q$ on $N^*\mathcal{F}$ with the coordinates $(x, y, \eta) \in I^p \times I^q \times \mathbb{R}^q$. This chart is a foliated chart on $N^*\mathcal{F}$ for the linearized foliation $\mathcal{F}_N$, and the restriction of $\mathcal{F}_N$ to U_1 is given by the level sets $y = \text{const}$, $\eta = \text{const}$.

The holonomy groupoid $G_{\mathcal{F}_N}$ of the linearized foliation $\mathcal{F}_N$ can be described as the set of all $(\gamma, \nu) \in G \times \tilde{N}^*\mathcal{F}$ such that $r(\gamma) = \pi(\nu)$. The source map $s_N : G_{\mathcal{F}_N} \to \tilde{N}^*\mathcal{F}$ and the range map $r_N : G_{\mathcal{F}_N} \to \tilde{N}^*\mathcal{F}$ are defined as $s_N(\gamma, \nu) = dh_\gamma^*(\nu)$ and $r_N(\gamma, \nu) = \nu$. We have a map $\pi_G : G_{\mathcal{F}_N} \to G$ given by $\pi_G(\gamma, \nu) = \gamma$.

The holonomy groupoid $G_{\mathcal{F}_N}$ carries a natural codimension q foliation $\mathcal{G}_N$. The leaf of $\mathcal{G}_N$ through a point $(\gamma, \nu) \in G_{\mathcal{F}_N}$ is the set of all $(\gamma', \nu') \in G_{\mathcal{F}_N}$ such that ν and ν' lie in the same leaf in $\mathcal{F}_N$. Let $|T\mathcal{G}_N|^{1/2}$ be the line bundle of leafwise half-densities on $G_{\mathcal{F}_N}$ with respect to the foliation $\mathcal{G}_N$. It is easy to see that

$$|T\mathcal{G}_N|^{1/2} = r_N^*(|T\mathcal{F}_N|^{1/2}) \otimes s_N^*(|T\mathcal{F}_N|^{1/2}),$$

where $s_N^*(|T\mathcal{F}_N|^{1/2})$ and $r_N^*(|T\mathcal{F}_N|^{1/2})$ denote the lifts of the line bundle $|T\mathcal{F}_N|^{1/2}$ of leafwise half-densities on $N^*\mathcal{F}$ via the source and the range mappings s_N and r_N respectively.

Let π^*E denote the lift of the vector bundle E to $\tilde{T}^*M = T^*M \setminus 0$ via the bundle map $\pi : \tilde{T}^*M \to M$. Denote by $\mathcal{L}(\pi^*E)$ the vector bundle on $G_{\mathcal{F}_N}$, whose fiber at a point $(\gamma, \nu) \in G_{\mathcal{F}_N}$ is the space $\mathcal{L}((\pi^*E)_{s_N(\gamma,\nu)}, (\pi^*E)_{r_N(\gamma,\nu)})$ of linear maps from $(\pi^*E)_{s_N(\gamma,\nu)}$ to $(\pi^*E)_{r_N(\gamma,\nu)}$.

A section $k \in C^\infty(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ is said to be properly supported if the restriction of the map $r : G_{\mathcal{F}_N} \to \tilde{N}^*\mathcal{F}$ to $\operatorname{supp} k$ is a proper map. Consider the space $C^\infty_{\text{prop}}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ of smooth, properly supported sections of $\mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2}$. One can introduce the structure of involutive algebra on $C^\infty_{\text{prop}}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ by the standard formulas. Let $S^m(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ be the space of all $s \in C^\infty_{\text{prop}}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ homogeneous of degree m with respect to the action of $\mathbb{R}$ given by the multiplication in the fibers of the vector bundle $\pi_G : G_{\mathcal{F}_N} \to G$.

Now let $\varkappa : U \subset M \to I^p \times I^q$, $\varkappa' : U' \subset M \to I^p \times I^q$ be two compatible foliated charts on M. Then the corresponding foliated charts $\varkappa_n : U_1 \subset N^*\mathcal{F} \to I^p \times I^q \times \mathbb{R}^q$, $\varkappa_n' : U_1' \subset N^*\mathcal{F} \to I^p \times I^q \times \mathbb{R}^q$ are compatible with respect to the foliation $\mathcal{F}_N$. So they define a foliated chart V on the foliated manifold $(G_{\mathcal{F}_N}, \mathcal{G}_N)$ with the coordinates $(x, x', y, \eta) \in I^p \times I^p \times I^q \times \mathbb{R}^q$, and the restriction of $\mathcal{G}_N$ to V is given by the level sets $y = \text{const}$, $\eta = \text{const}$.

The principal symbol $\sigma(A)$ of an operator $A \in \Psi^{m,-\infty}(M, \mathcal{F}, E)$ given in local coordinates by the formula (2) is globally defined as an element of the space $S^m(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$. Thus, we have the half-density principal symbol mapping

$$\sigma : \Psi^{m,-\infty}(M, \mathcal{F}, E) \to S^m(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2}), \tag{3}$$

which satisfies

$$\sigma(AB) = \sigma(A)\sigma(B), \qquad \sigma(A^*) = \sigma(A)^*$$

for any $A \in \Psi^{m_1,-\infty}(M, \mathcal{F}, E)$ and $B \in \Psi^{m_2,-\infty}(M, \mathcal{F}, E)$.

Example 2.1. Suppose that a foliation $\mathcal{F}$ on a compact manifold M is given by the fibers of a fibration $f : M \to B$ over a compact manifold B. Then, for any $x \in M$, $N_x^*\mathcal{F}$ coincides with the image of the cotangent map $f^* : T^*_{f(x)}B \to T^*_xM$. The inverse map $(f^*)^{-1} : T^*_xM \to T^*_{f(x)}B$ is a fibration whose fibers are the leaves of the linearized foliation $\mathcal{F}_N$. Thus, we have the diffeomorphism

$$\{(x, \xi) \in M \times T^*B : f(x) = \pi_B(\xi)\} \xrightarrow{\cong} N^*\mathcal{F}, \quad (x, \xi) \mapsto f^*(\xi) \in N_x^*\mathcal{F},$$

where $\pi_B : T^*B \to B$ is the cotangent bundle map. So the diagram

$$\begin{array}{ccc} N^*\mathcal{F} & \xrightarrow{\ \pi\ } & M \\ {\scriptstyle (f^*)^{-1}}\downarrow & & \downarrow{\scriptstyle f} \\ T^*B & \xrightarrow[\pi_B]{} & B \end{array}$$

commutes, and $N^*\mathcal{F}$ can be considered as the pull back of the bundle $f : M \to B$ to T^*B:

$$N^*\mathcal{F} \cong \pi_B^*(M) = \{(x, \xi) \in M \times T^*B : f(x) = \pi(\xi)\}. \tag{4}$$

The holonomy groupoid G of $\mathcal{F}$ is the fiber product

$$M \times_B M = \{(x, y) \in M \times M : f(x) = f(y)\},$$

where $s(x, y) = y$, $r(x, y) = x$. The holonomy groupoid $G_{\mathcal{F}_N}$ can be identified as above with

$$N^*\mathcal{F} \times_{T^*B} N^*\mathcal{F} \cong \{(x, y, \xi) \in M \times M \times T^*B : f(x) = f(y) = \pi_B(\xi)\},$$

where $s_N(x,y,\xi)=y$, $r_N(x,y,\xi)=x$. So we have the commutative diagram

$$\begin{array}{ccc} G_{\mathcal{F}_N} & \xrightarrow{\pi_G} & G \\ \downarrow & & \downarrow \\ T^*B & \xrightarrow[\pi_B]{} & B \end{array}$$

and the foliation $\mathcal{G}_N$ is given by the fibers of the fibration $G_{\mathcal{F}_N}\to T^*B$.

For any $\xi\in T^*B$, let $\Psi^{-\infty}((N^*\mathcal{F})_\xi,(\pi^*E)_\xi)$ be the involutive algebra of all smoothing operators, acting on the space of smooth half-densities $C^\infty((N^*\mathcal{F})_\xi,(\pi^*E)_\xi)$, where $(N^*\mathcal{F})_\xi$ is the fiber of the fibration $N^*\mathcal{F}\to T^*B$ at ξ and $(\pi^*E)_\xi$ is the restriction of π^*E to $(N^*\mathcal{F})_\xi$. Consider a sheaf $\Psi^{-\infty}(N^*\mathcal{F},\pi^*E)$ of involutive algebras on T^*B whose stalk at $\xi\in T^*B$ is $\Psi^{-\infty}((N^*\mathcal{F})_\xi,(\pi^*E)_\xi)$. For any section σ of the sheaf $\Psi^{-\infty}(N^*\mathcal{F},\pi^*E)$, the Schwartz kernels of the operators $\sigma(\xi)$ determine a well-defined section of $\mathcal{L}(\pi^*E)\otimes|T\mathcal{G}_N|^{1/2}$ over $G_{\mathcal{F}_N}\cong N^*\mathcal{F}\times_{T^*B}N^*\mathcal{F}$. We say that σ is smooth if the corresponding section is smooth. This defines an algebra isomorphism of $\Psi^{-\infty}(N^*\mathcal{F},\pi^*E)$ with $C^\infty(G_{\mathcal{F}_N},\mathcal{L}(\pi^*E)\otimes|T\mathcal{G}_N|^{1/2})$.

Remark 2.2. Suppose as above that a foliation $\mathcal{F}$ on a compact manifold M is given by the fibers of a fibration $f: M\to B$ over a compact manifold B. If we consider the cotangent bundle T^*M as a symplectic manifold equipped with the canonical symplectic structure, then $N^*\mathcal{F}$ is a closed coisotropic submanifold, and the linearized foliation $\mathcal{F}_N$ coincides with the null-foliation of this coisotropic submanifold, that is, $T\mathcal{F}_N$ is the skew-orthogonal complement of $T(N^*\mathcal{F})$ in $T(T^*M)$. It is well known that the fiber product $N^*\mathcal{F}\times_{T^*B}N^*\mathcal{F}$ is a canonical relation in T^*M, which is often called the flowout of the coisotropic submanifold $N^*\mathcal{F}$. The algebra of Fourier integral operators associated with this canonical relation was introduced in [14]. In this particular case, it coincides with the algebra $\Psi^{*,-\infty}(M,\mathcal{F},E)$.

For an arbitrary compact foliated manifold $(M,\mathcal{F})$, one can consider $G_{\mathcal{F}_N}$ as an immersed canonical relation in T^*M, and the associated algebra of Fourier integral operators also coincides with $\Psi^{*,-\infty}(M,\mathcal{F},E)$. One has to be a little bit careful, defining the algebra of Fourier integral operators associated with an immersed canonical relation (see [21] for more details).

3. Transverse principal and subprincipal symbols

Recall that the principal symbol of an operator $P\in\Psi^m(M,E)$ is an element of the space $S^m(\tilde{T}^*M,\mathrm{End}\,(\pi^*E))$ of smooth sections of the vector bundle $\mathrm{End}\,(\pi^*E)$, homogeneous of degree m with respect to the $\mathbb{R}$-multiplication in the fibers of $\mathrm{End}\,(\pi^*E)$.

By definition, the transversal principal symbol $\sigma(P)$ of $P\in\Psi^m(M,E)$ is the restriction of its principal symbol to $\tilde{N}^*\mathcal{F}$. So we have

$$\sigma(P)\in S^m(\tilde{N}^*\mathcal{F},\mathrm{End}\,(\pi^*E)).$$

The principal symbol of P in a foliated chart is given by the top degree homogeneous component p_m of its complete symbol p, and the transverse principal symbol is given by

$$\sigma(P)(x,y,\eta)=p_m(x,y,0,\eta),\quad (x,y,\eta)\in I^p\times I^q\times\mathbb{R}^q.$$

Before passing to the definition of the transverse subprincipal symbol, we recall the concept of a partial connection.

By a partial connection on a vector bundle V over a smooth manifold X along a vector field v on X we will understand a linear map $\nabla_v: C^\infty(X,V)\to C^\infty(X,V)$ satisfying

$$\nabla_v(fs)=v(f)s+f\nabla_v(s),\quad f\in C^\infty(X),\ s\in C^\infty(X,V).$$

If we fix a trivialization of V over an open subset $U\subset M$, then one can write

$$\nabla_v=v\cdot Id+\Gamma$$

on $C^\infty(U,\mathbb{C}^N)$ for some $\Gamma\in C^\infty(U,\mathrm{End}(\mathbb{C}^N))$. Under a change of trivializations by a function $T\in C^\infty(U,\mathrm{GL}\,(N,\mathbb{C}))$, we get the transformation law

$$\Gamma'=T^{-1}\Gamma T+T^{-1}v(T).$$

Let $f_t : X \to X$ be the flow on X generated by v. One can define the parallel transport on V along the orbits of v as follows. Let $x \in X$ and $w \in V_x$. Let the section $\tau \in [0, t] \mapsto w(\tau) \in V_{f_\tau(x)}$ be a solution in local coordinates of the Cauchy problem

$$\frac{\mathrm{d}w(\tau)}{\mathrm{d}\tau} = \Gamma(f_\tau(x)), \quad \tau \in [0, t],$$
$$v(0) = w.$$

The parallel transport of w along the orbit $\{f_\tau(x) : \tau \in [0, t]\}$ is defined as $\alpha_t(w) = w(t) \in V_{f_t(x)}$.

The parallel transport determines a flow α_t on the vector bundle V which projects to the flow f_t on X under the bundle map $V \to X$ and makes V an $\mathbb{R}$-equivariant vector bundle.

The induced flow α_t^* on $C^\infty(X, V)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_t^* s = \nabla_v(\alpha_t^* s), \quad s \in C^\infty(X, V). \tag{5}$$

Now we go back to the foliation setting. Assume that an operator $P \in \Psi^m(M, E)$ has the scalar and real principal symbol $p_m \in C^\infty(\tilde{T}^*M)$. Let X_{p_m} be the Hamiltonian vector field of p_m. Recall that, in a foliation chart, X_{p_m} is given by

$$X_{p_m} = \sum_{j=1}^{p}\left(\partial_{\xi_j} p_m \frac{\partial}{\partial x_j} - \partial_{x_j} p_m \frac{\partial}{\partial \xi_j}\right) + \sum_{k=1}^{q}\left(\partial_{\eta_k} p_m \frac{\partial}{\partial y_k} - \partial_{y_k} p_m \frac{\partial}{\partial \eta_k}\right).$$

As in [16], we define the subprincipal symbol of an operator $P \in \Psi^m(M, E)$ as a partial connection $\nabla_{\mathrm{sub}}(P)$ on π^*E along the Hamiltonian vector field X_{p_m}. In local coordinates, we have

$$\nabla_{\mathrm{sub}}(P) = X_{p_m} + \mathrm{i}p_{\mathrm{sub}},$$

where

$$p_{\mathrm{sub}}(x, y, \xi, \eta) = p_{m-1}(x, y, \xi, \eta) - \frac{1}{2\mathrm{i}}\sum_{j=1}^{p}\frac{\partial^2 p_m}{\partial x_j \partial \xi_j}(x, y, \xi, \eta) - \frac{1}{2\mathrm{i}}\sum_{l=1}^{q}\frac{\partial^2 p_m}{\partial y_l \partial \eta_l}(x, y, \xi, \eta).$$

Now in addition assume that the transverse principal symbol of $P \in \Psi^m(M, E)$ is holonomy invariant. A function $\sigma \in C^\infty(\tilde{N}^*\mathcal{F})$ is called holonomy invariant if it satisfies the following condition:

$$\sigma(\mathrm{d}h_\gamma^*(\nu)) = \sigma(\nu), \gamma \in G, \nu \in N^*_{r(\gamma)}\mathcal{F}.$$

In a foliation chart, holonomy invariance of σ means that

$$\sigma(x, y, \eta) = \sigma(y, \eta), \quad (x, y, \eta) \in I^p \times I^q \times \mathbb{R}^q.$$

Observe also that $\sigma \in C^\infty(\tilde{N}^*\mathcal{F})$ is holonomy invariant if and only if it is constant along the leaves of $\mathcal{F}_N$.

Under these assumptions, the vector field X_{p_m} is tangent to $N^*\mathcal{F}$. We define the transverse subprincipal symbol of P as the restriction of its subprincipal symbol to $\tilde{N}^*\mathcal{F}$. In a foliated coordinate chart, it is given by

$$\nabla_{\mathrm{sub}}(P) = X_{p_m} + \mathrm{i}\sigma_{\mathrm{sub}}(P),$$

where

$$\sigma_{\mathrm{sub}}(P)(x, y, \eta) = p_{m-1}(x, y, 0, \eta) - \frac{1}{2\mathrm{i}}\sum_{j=1}^{p}\frac{\partial^2 p_m}{\partial x_j \partial \xi_j}(x, y, 0, \eta) - \frac{1}{2\mathrm{i}}\sum_{l=1}^{q}\frac{\partial^2 \sigma(P)}{\partial y_l \partial \eta_l}(y, \eta). \tag{6}$$

4. A $\Psi^*(M, E)$-bimodule structure

In this section, we will study the structure of a $\Psi^*(M, E)$-bimodule on the algebra $\Psi^{*,-\infty}(M, \mathcal{F}, E)$ given by the composition of operators. An important new statement is the corresponding formula for the complete symbols, which is given in the following theorem.

Theorem 4.1. *If A is given by (1) with some $k_A \in S^{m_1}(I^p \times I^p \times I^q \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^r))$ and $B \in \Psi^{m_2}(I^n, \mathbb{C}^r)$, then AB and BA are given by (1) with some $k_{AB} \in S^{m_1+m_2}(I^p \times I^p \times I^q \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^r))$ and $k_{BA} \in S^{m_1+m_2}(I^p \times I^p \times I^q \times \mathbb{R}^q, \mathcal{L}(\mathbb{C}^r))$, which admit the following asymptotic expansions:*

$$k_{AB}(x, x', y, \eta) \sim \sum_{\alpha,\beta} \frac{1}{\alpha!\beta!} \partial_\xi^\alpha \partial_\eta^\beta b(x, y, 0, \eta) D_x^\alpha D_y^\beta k_A(x, x', y, \eta),$$

$$k_{BA}(x, x', y, \eta) \sim \sum_{\alpha,\beta} \frac{1}{\alpha!\beta!} D_{x'}^\alpha \partial_\eta^\beta k_A(x, x', y, \eta)(-\partial_\xi)^\alpha D_y^\beta b(x', y, 0, \eta).$$

The proof of this theorem can be achieved by a straightforward modification of the standard arguments. As an immediate consequence, we get:

Proposition 4.2 (*[21]*). *If $A \in \Psi^{m_1,-\infty}(M, \mathcal{F}, E)$ and $B \in \Psi^{m_2}(M, E)$, then AB and BA in $\Psi^{m_1+m_2,-\infty}(M, \mathcal{F}, E)$ and*

$$\sigma(AB) = \sigma(A) \cdot r_N^* \sigma(B), \qquad \sigma(BA) = s_N^* \sigma(B) \cdot \sigma(A).$$

Now we assume that $B \in \Psi^{m_2}(M, E)$ is such that the principal symbol of is real and scalar, and its transverse principal symbol is holonomy invariant. By Proposition 4.2, it follows that, for any $A \in \Psi^{m_1}(M, \mathcal{F}, E)$, the operator $[A, B]$ belongs to $\Psi^{m_1+m_2-1,-\infty}(M, \mathcal{F}, E)$. Using Theorem 4.1, one can compute the principal symbol of $[A, B]$.

Denote by b_{m_2} the principal symbol of B. As above, X_b denotes the restriction of the Hamiltonian vector field of b_{m_2} to $N^*\mathcal{F}$. Since X_b is an infinitesimal transformation of $\mathcal{F}_N$, there exists a vector field $\mathcal{H}_b$ on $G_{\mathcal{F}_N}$ such that $\mathrm{d}s_N(\mathcal{H}_b) = X_b$ and $\mathrm{d}r_N(\mathcal{H}_b) = X_b$. In local coordinates, $\mathcal{H}_b$ is given by

$$\begin{aligned}\mathcal{H}_b(x, x', y, \eta) = {} & \sum_{j=1}^p \partial_{\xi_j} b_{m_2}(x, y, 0, \eta) \frac{\partial}{\partial x_j} + \sum_{j=1}^p \partial_{\xi_j} b_{m_2}(x', y, 0, \eta) \frac{\partial}{\partial x'_j} \\ & + \sum_{k=1}^q \left(\partial_{\eta_k} \sigma_B(y, \eta) \frac{\partial}{\partial y_k} - \partial_{y_k} \sigma_B(y, \eta) \frac{\partial}{\partial \eta_k} \right), \quad (x, x', y, \eta) \in I^p \times I^p \times I^q \times \mathbb{R}^q.\end{aligned}$$

Denote by $\mathcal{L}_{\mathcal{H}_b}$ the Lie derivative on the space $C^\infty_{\mathrm{prop}}(G_{\mathcal{F}_N}, |T\mathcal{G}_N|^{1/2})$ by the vector field $\mathcal{H}_b$. In a foliated chart, it defines a derivative on the space $C^\infty_{\mathrm{prop}}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^* E) \otimes |T\mathcal{G}_N|^{1/2})$. For any $k \in C^\infty_{\mathrm{prop}}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^* E) \otimes |T\mathcal{G}_N|^{1/2})$ of the form $k = k(x, x', y, \eta)|\mathrm{d}x|^{1/2}|\mathrm{d}x'|^{1/2}$, we have

$$\begin{aligned}\mathcal{L}_{\mathcal{H}_b} k = {} & \left(\mathcal{H}_b k(x, x', y, \eta) + \frac{1}{2} \sum_{j=1}^p D_{x_j} \partial_{\xi_j} b_{m_2}(x, y, 0, \eta) k(x, x', y, \eta) \right. \\ & \left. - \frac{1}{2} \sum_{j=1}^p D_{x'_j} \partial_{\xi_j} b_{m_2}(x', y, 0, \eta) k(x, x', y, \eta) \right) |\mathrm{d}x|^{1/2} |\mathrm{d}x'|^{1/2}, \\ & (x, x', y, \eta) \in I^p \times I^p \times I^q \times \mathbb{R}^q.\end{aligned}$$

The transverse subprincipal symbol of the operator B considered as a partial connection on $N^*\mathcal{F}$ along X_b yields the corresponding partial connection on the space $C^\infty_{\mathrm{prop}}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^* E) \otimes |T\mathcal{G}_N|^{1/2})$ along $\mathcal{H}_b$. In a foliation chart, for any $k \in C^\infty_{\mathrm{prop}}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^* E) \otimes |T\mathcal{G}_N|^{1/2})$, we have

$$\nabla_{\mathcal{H}_b} k = \mathcal{L}_{\mathcal{H}_b} k + \mathrm{i}(k \cdot r_N^* \sigma_{\mathrm{sub}}(B) - s_N^* \sigma_{\mathrm{sub}}(B) \cdot k).$$

By a straightforward calculation, Theorem 4.1 implies the following result.

Theorem 4.3. *Let $A \in \Psi^{m_1}(M, \mathcal{F}, E)$ and $B \in \Psi^{m_2}(M, E)$. Suppose that the principal symbol of B is real and scalar, and the transverse principal symbol of B is holonomy invariant. Then*

$$\sigma([B, A]) = \frac{1}{\mathrm{i}} \nabla_{\mathcal{H}_b} \sigma(A). \tag{7}$$

5. Transverse bicharacteristic flow

In this section, we give a definition of the transverse bicharacteristic flow associated with a first-order transversally elliptic operator.

Consider an operator $P \in \Psi^1(M, E)$ which has the real scalar principal symbol and the holonomy invariant transverse principal symbol. Let $p \in S^1(\tilde{T}^*M)$ be the principal symbol of P. The Hamiltonian flow f_t of p preserves $\tilde{N}^*\mathcal{F}$, and its restriction to $N^*\mathcal{F}$ (denoted also by f_t) preserves the foliation $\mathcal{F}_N$, that is, takes any leaf of $\mathcal{F}_N$ to a leaf. Moreover, one can show that there exists a flow F_t on $G_{\mathcal{F}_N}$ such that $s_N \circ F_t = f_t \circ s_N$, $r_N \circ F_t = f_t \circ r_N$, which preserves the foliation $\mathcal{G}_N$. Actually, this flow is generated by the vector field $\mathcal{H}_p$ introduced in Section 4. It is easy to see that the flow F_t depends only on the 1-jet of the principal symbol of P along $N^*\mathcal{F}$.

Let α_t^* be the flow on $C^\infty(N^*\mathcal{F}, \pi^*E)$ determined by the subprincipal symbol $\nabla^{\mathrm{sub}}(P)$ of P (see (5)). It induces the flow $\mathrm{Ad}(\alpha_t)^*$ on the space $C^\infty_{\mathrm{prop}}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$, which satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ad}(\alpha_t)^* k = \nabla_{\mathcal{H}_p} k, \quad k \in C^\infty_{\mathrm{prop}}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2}).$$

This flow will be called the transverse bicharacteristic flow of P. One can show that

$$\mathrm{Ad}(\alpha_t)^* \circ s_N^* = s_N^* \circ \alpha_t^*, \qquad \mathrm{Ad}(\alpha_t)^* \circ r_N^* = r_N^* \circ \alpha_t^*.$$

Now consider a transversally elliptic operator $A \in \Psi^2(M, E)$, which has the positive scalar principal symbol and the holonomy invariant transverse principal symbol. (Recall that an operator $P \in \Psi^m(M, E)$ is said to be transversally elliptic if $\sigma_P(\nu)$ is invertible for any $\nu \in \tilde{N}^*\mathcal{F}$.) Let $a_2 \in S^2(\tilde{T}^*M)$ be the principal symbol of A: $a_2 \geq 0$. Then the operator $\sqrt{A}$ is not, in general, well defined, and even if A is positive self-adjoint and the operator $\sqrt{A}$ is a well-defined positive operator in $L^2(M, E)$, it is not, in general, a pseudodifferential operator. Nevertheless, we can define its transverse bicharacteristic flow, working at the level of symbols.

By assumption, a_2 is positive in some conic neighborhood of $\tilde{N}^*\mathcal{F}$. Take any scalar elliptic symbol $\tilde{p} \in S^1(\tilde{T}^*M)$ which is equal to $\sqrt{a_2}$ in some conic neighborhood of $\tilde{N}^*\mathcal{F}$ (indeed, it is sufficient that the 1-jets of $\tilde{p}$ and $\sqrt{a_2}$ coincide on $\tilde{N}^*\mathcal{F}$). Proceeding as above, we obtain the flow $\mathrm{Ad}(\alpha_t)^*$ on $C^\infty_{\mathrm{prop}}(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$, which is independent of a choice of $\tilde{p}$ and will be called the transverse bicharacteristic flow of $\sqrt{A}$.

Example 5.1. Suppose that $\mathcal{F}$ is a Riemannian foliation and g_M is a bundle-like metric on M. Recall that a Riemannian metric g_M on M is bundle-like if the induced metric on the normal bundle $Q = TM/T\mathcal{F}$ is holonomy invariant, that is, for any continuous leafwise path γ from x to y, the corresponding linear holonomy map $\mathrm{d}h_\gamma : Q_x \to Q_y$ is an isometry (see, for instance, [25,26] for more details on Riemannian foliations).

For any $x \in M$, let $T^H_x M = T_x\mathcal{F}^\perp$. So we have a smooth vector subbundle $T^H M$ of TM such that

$$TM = T^H M \oplus T\mathcal{F}. \tag{8}$$

There is a natural isomorphism $T^H M \cong Q$. Observe also natural isomorphisms $T^H M^* \cong Q^* \cong N^*\mathcal{F}$.

The decomposition (8) induces a bigrading on ΛT^*M:

$$\Lambda^k T^*M = \bigoplus_{i=0}^{k} \Lambda^{i,k-i} T^*M, \quad k = 0, 1, \ldots, n,$$

where $\Lambda^{i,j} T^*M = \Lambda^i T\mathcal{F}^* \otimes \Lambda^j T^H M^*$. In this bigrading, the de Rham differential d can be written as

$$\mathrm{d} = \mathrm{d}_F + \mathrm{d}_H + \theta,$$

where d_F and d_H are first-order differential operators (called the tangential de Rham differential and the transversal de Rham differential respectively), and θ is a zero-order differential operator.

By definition, the transverse signature operator is a first-order differential operator in $C^\infty(M, \Lambda T^H M^*)$ given by

$$D_H = \mathrm{d}_H + \mathrm{d}_H^*.$$

The principal symbol of D_H^2 (see Theorem 9.2 below) is given by

$$a_2(x, \xi) = g^M(P^H(x, \xi), P^H(x, \xi)), \quad (x, \xi) \in T^*M,$$

where g^M is the induced metric on T^*M, $P^H : T^*M \to T^HM^*$ is the orthogonal projection. The holonomy invariance of the transverse principal symbol is equivalent to the bundle-like property of the metric.

The transverse bicharacteristic flow of the operator $\langle D_H \rangle = (D^2 + I)^{1/2}$ coincides with the transverse geodesic flow γ_t^M of g_M, which is the restriction of the geodesic flow of g_M to $N^*\mathcal{F}$.

Example 5.2. Suppose that a foliation $\mathcal{F}$ on a compact manifold M is given by the fibers of a fibration $f : M \to B$ over a compact manifold B. A Riemannian metric g_M on M is bundle-like if and only if there exists a Riemannian metric g_B on B such that, for any $x \in M$, the tangent map f_* induces an isometry from $(T_x^H M, g_M|_{T^HM})$ to $(T_{f(x)}B, g_B)$, or, equivalently, $f : (M, g_M) \to (B, g_B)$ is a Riemannian submersion. Then the transverse geodesic flow γ_t^M of g_M projects under the map f^* to the geodesic flow γ_t^B of g_B that implies commutativity of the following diagram:

$$\begin{array}{ccc} N^*\mathcal{F} & \xrightarrow{\gamma_t^M} & N^*\mathcal{F} \\ f^*\big\uparrow & & \big\uparrow f^* \\ T^*B & \xrightarrow{\gamma_t^B} & T^*B \end{array}$$

Commutativity of this diagram allows us to lift the flow γ_t^M to the holonomy groupoid $G_{\mathcal{F}_N} \cong N^*\mathcal{F} \times_{T^*B} N^*\mathcal{F}$ as above.

Example 5.3. Suppose that, in the setting of the previous example, the fibration $f : M \to B$ is a principal K-bundle with a compact group K. The group K has a natural Hamiltonian action on the cotangent bundle T^*M. The conormal bundle $N^*\mathcal{F}$ is a K-invariant submanifold of T^*M, and the fibration $(f^*)^{-1} : N^*\mathcal{F} \to T^*B$ is a principal K-bundle.

Suppose that ω is a connection on the principal bundle $f : M \to B$. It gives rise to a decomposition

$$T_mM = V_m \oplus H_m, \quad m \in M,$$

where V_m is the vertical space and H_m is the connection's horizontal distribution. The vertical space V_m is naturally isomorphic to the Lie algebra $\mathfrak{k}$ of K, and the horizontal space H_m is identified with the tangent space $T_{f(m)}B$ of the base. Choose a Riemannian metric on B and a bi-invariant metric on K, and define a K-invariant Riemannian metric on M, by requiring that, on V_m, it is induced by the fixed bi-invariant metric on K, on H_m, it is the lift of the Riemannian metric on B, and V_m and H_m are orthogonal. Such a metric is sometimes called the Kaluza–Klein metric of the connection. The fibers of the bundle $f : M \to B$ are totally geodesic submanifolds, which are isometric to K.

The pull back of the connection form ω on M defines a connection form $\pi_M^*\omega$ on the principal bundle $(f^*)^{-1} : N^*\mathcal{F} \to T^*B$. The transverse geodesic flow γ_t^M of the Kaluza–Klein metric is described as follows. For any $\nu \in N_m^*\mathcal{F}$, the element $\gamma_t^M(\nu) \in N^*\mathcal{F}$ is obtained by the parallel transport of ν along the orbit $\{\gamma_\tau^B((f^*)^{-1}(\nu)) : \tau \in [0, t]\}$ of the geodesic flow γ_t^B on T^*B with respect to the connection $\pi_B^*\omega$.

Example 5.4. Now suppose that a fibration $f : M \to B$ as above is the orthonormal frame bundle $F(B) \to B$ of the Riemannian manifold B. So, for any $x \in B$, the fiber $F(B)_x$ consists of all orthonormal frames $(v_1, v_2, \ldots, v_q)$ in T_xB. It is a principal bundle with structure group $O(q)$. The Riemannian metric on B gives rise to a natural (Levi-Civita) connection on $f : F(B) \to B$. Fix a bi-invariant Riemannian metric on $O(q)$ and consider the corresponding Kaluza–Klein metric on $F(B)$.

By (4), it follows that

$$N^*\mathcal{F} \cong \{((v_1, v_2, \ldots, v_q), \xi) \in F(B)_x \times T_x^*B : x \in B\}.$$

For any $((v_1, v_2, \ldots, v_q), \xi) \in N^*\mathcal{F}$, the action of the transverse geodesic flow γ_t^M is described as

$$\gamma_t^M((v_1, v_2, \ldots, v_q), \xi) = ((v_1(t), v_2(t), \ldots, v_q(t)), \xi(t)),$$

where $\xi(t) = \gamma_t^B(\xi)$ and, for any $i = 1, \ldots, q$, the vector $v_i(t)$ is obtained by the parallel transport of v_i along the geodesic $\{\pi(\gamma_\tau^B(\xi)) : \tau \in [0, t]\}$ with respect to the Levi-Civita connection on TB. Since $\xi(t) = \gamma_t^B(\xi)$ can obtained by the parallel transport of ξ along the geodesic $\{\pi(\gamma_\tau^B(\xi)) : \tau \in [0, t]\}$ with respect to the Levi-Civita connection on T^*B, the transverse geodesic flow γ_t^M has q first integrals $I_1, I_2, \ldots, I_q \in C^\infty(N^*\mathcal{F})$ given by

$$I_j((v_1, v_2, \ldots, v_q), \xi) = \xi(v_j), \quad j = 1, \ldots, q.$$

There is a natural global right action of the group $SO(q)$ in the fibers of the bundle

$$N^*\mathcal{F} \to T^*B, \quad ((v_1, v_2, \dots, v_q), \xi) \in N^*\mathcal{F} \mapsto \xi \in T^*B.$$

For every orthogonal matrix $A = (a_{ij}) \in SO(q)$ and any $((v_1, v_2, \dots, v_q), \xi) \in N^*\mathcal{F}$ we put

$$A((v_1, v_2, \dots, v_q), \xi) = \left(\left(\sum_{i=1}^q v_i a_{i1}, \sum_{i=1}^q v_i a_{i2}, \dots, \sum_{i=1}^q v_i a_{iq}\right), \xi\right).$$

This action obviously commutes with the transverse geodesic flow γ_t^M. Moreover, we have

$$I_j(A(v_1, v_2, \dots, v_q), \xi) = \sum_{i=1}^q I_i((v_1, v_2, \dots, v_q), \xi) a_{ij}.$$

Therefore, the restrictions of the transverse geodesic flow γ_t^M to the level sets $(N^*\mathcal{F})_c$ defined by

$$I_j((v_1, v_2, \dots, v_q), \xi) = c_j, \quad j = 1, 2, \dots, q,$$

are isomorphic for different values of $c = (c_1, c_2, \dots, c_q) \in \mathbb{R}^q$. It is easy to see that, for any $c \in \mathbb{R}^q$, $(N^*\mathcal{F})_c$ can be identified with the frame bundle $F(B)$, and, for $c = (1, 0, \dots, 0)$, the restriction of γ_t^M to $(N^*\mathcal{F})_c$ is precisely the frame flow on $F(B)$ (see [16] and references therein).

6. Egorov's theorem

Let $D \in \Psi^1(M, E)$ be a formally self-adjoint, transversally elliptic operator such that D^2 has the scalar principal symbol and the holonomy invariant transverse principal symbol. By [21], the operator D is essentially self-adjoint with initial domain $C^\infty(M, E)$. Define an unbounded linear operator $\langle D\rangle$ in the space $L^2(M, E)$ as

$$\langle D\rangle = (D^2 + I)^{1/2}.$$

By the spectral theorem, the operator $\langle D\rangle$ is well defined as a positive, self-adjoint operator in $L^2(M, E)$. It can be shown that $H^1(M, E)$ is contained in the domain of $\langle D\rangle$ in $L^2(M, E)$.

By the spectral theorem, the operator $\langle D\rangle^s = (D^2 + I)^{s/2}$ is a well-defined positive self-adjoint operator in $\mathcal{H} = L^2(M, E)$ for any $s \in \mathbb{R}$, which is unbounded if $s > 0$. For any $s \geq 0$, denote by $\mathcal{H}^s$ the domain of $\langle D\rangle^s$, and, for $s < 0$, $\mathcal{H}^s = (\mathcal{H}^{-s})^*$. Put also $\mathcal{H}^\infty = \bigcap_{s\geq 0} \mathcal{H}^s$, $\mathcal{H}^{-\infty} = (\mathcal{H}^\infty)^*$. It is clear that $H^s(M, E) \subset \mathcal{H}^s$ for any $s \geqslant 0$ and $\mathcal{H}^s \subset H^s(M, E)$ for any $s < 0$. In particular, $C^\infty(M, E) \subset \mathcal{H}^s$ for any s.

We say that a bounded operator A in $\mathcal{H}^\infty$ belongs to $\mathcal{L}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ (resp. $\mathcal{K}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$), if, for any s and r, it extends to a bounded (resp. compact) operator from $\mathcal{H}^s$ to $\mathcal{H}^r$, or, equivalently, the operator $\langle D\rangle^r A \langle D\rangle^{-s}$ extends to a bounded (resp. compact) operator in $L^2(M, E)$. It is easy to see that $\mathcal{L}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ is a involutive subalgebra in $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ is its ideal. We also introduce the class $\mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$, which consists of all operators from $\mathcal{K}(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$ such that, for any s and r, the operator $\langle D\rangle^r A \langle D\rangle^{-s}$ is a trace class operator in $L^2(M, E)$. It should be noted that any operator K with the smooth kernel belongs to $\mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$.

By the spectral theorem, the operator $\langle D\rangle$ defines a strongly continuous group $\mathrm{e}^{\mathrm{i}t\langle D\rangle}$ of bounded operators in $L^2(M, E)$. Consider a one-parameter group Φ_t of $*$-automorphisms of the algebra $\mathcal{L}(L^2(M, E))$ defined by

$$\Phi_t(T) = \mathrm{e}^{\mathrm{i}t\langle D\rangle} T \mathrm{e}^{-\mathrm{i}t\langle D\rangle}, \quad T \in \mathcal{L}(L^2(M, E)),\ t \in \mathbb{R}.$$

The main result of the paper is the following theorem.

Theorem 6.1. *Let $D \in \Psi^1(M, E)$ be a formally self-adjoint, transversally elliptic operator such that D^2 has the scalar principal symbol and the holonomy invariant transverse principal symbol. For any $K \in \Psi^{m,-\infty}(M, \mathcal{F}, E)$, there exists an operator $K(t) \in \Psi^{m,-\infty}(M, \mathcal{F}, E)$ such that $\Phi_t(K) - K(t)$, $t \in \mathbb{R}$, is a smooth family of operators of class $\mathcal{L}^1(\mathcal{H}^{-\infty}, \mathcal{H}^\infty)$.*

Moreover, if $k \in S^m(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ is the principal symbol of K, then the principal symbol $k_t \in S^m(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ of the operator $K(t)$ is given by

$$k_t = \mathrm{Ad}(\alpha_t)^*(k), \tag{9}$$

where $\mathrm{Ad}(\alpha_t)^*$ is the transverse bicharacteristic flow of the operator $\langle D\rangle$.

Proof. Let $\mathcal{L}(\mathcal{D}'(M,E),\mathcal{H}^\infty)$ (resp. $\mathcal{L}(\mathcal{H}^{-\infty},C^\infty(M,E))$) be the space of all bounded operators from $\mathcal{D}'(M,E)$ to $\mathcal{H}^\infty$ (resp. from $\mathcal{H}^{-\infty}$ to $C^\infty(M,E)$). Since any operator from $\Psi^{-N}(M,E)$ with $N > \dim M$ is a trace class operator in $L^2(M,E)$, one can easily see that $\mathcal{L}(\mathcal{D}'(M,E),\mathcal{H}^\infty)\subset\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^\infty)$ and $\mathcal{L}(\mathcal{H}^{-\infty},C^\infty(M,E))\subset\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^\infty)$.

As shown in [23], the operator $\langle D\rangle=(D^2+I)^{1/2}$ can be written as $\langle D\rangle = P+R$, where $P\in\Psi^1(M,E)$ is a self-adjoint, elliptic operator with the positive, scalar principal symbol and the holonomy invariant transversal principal symbol, and, for any $K\in\Psi^{*,-\infty}(M,\mathcal{F},E)$, $KR\in\mathcal{L}(\mathcal{H}^{-\infty},C^\infty(M,E))$ and $RK\in\mathcal{L}(\mathcal{D}'(M,E),\mathcal{H}^\infty)$.

Denote by e^{itP} the strongly continuous group of bounded operators in $L^2(M,E)$ generated by the elliptic operator iP. For $K\in\Psi^{m,-\infty}(M,\mathcal{F},E)$, let $\Phi^P_t(K)=e^{itP}Ke^{-itP}$. It is shown in [23] that the operator $\Phi^P_t(K)=e^{itP}Ke^{-itP}$ is in $\Psi^{m,-\infty}(M,\mathcal{F},E)$, and $\Phi_t(K)-\Phi^P_t(K)$, $t\in\mathbb{R}$, is a smooth family of operators of class $\mathcal{L}^1(\mathcal{H}^{-\infty},\mathcal{H}^\infty)$. So we can take $K(t)=\Phi^P_t(K)$. It remains to compute the principal symbol of $\Phi^P_t(K)$.

Without loss of generality, one can assume that the elliptic extension $\tilde{p}$ of p introduced in Section 5 to define the transverse bicharacteristic flow coincides with the principal symbol of P. We have

$$\frac{d}{dt}\Phi^P_t(K)=[iP,\Phi^P_t(K)],\quad t\in\mathbb{R},\ \Phi^P_0(K)=K.$$

Recall (cf. (5)) that the function k_t given by (9) satisfies the following equation:

$$\frac{d}{dt}k_t=\nabla_{\mathcal{H}_p}k_t. \tag{10}$$

Let $K_0(t)$ be any operator from $\Psi^{m,-\infty}(M,\mathcal{F},E)$ with the principal symbol k_t. Then, by (7) and (10), it follows that

$$\begin{aligned}&\frac{d}{dt}K_0(t)=[iP,K_0(t)]+R(t),\quad t\in\mathbb{R},\\ &K_0(0)=K+R_0\end{aligned}$$

where $R(t)\in\Psi^{m-1,-\infty}(M,\mathcal{F},E)$, $t\in\mathbb{R}$, and $R_0\in\Psi^{m-1,-\infty}(M,\mathcal{F},E)$. It is easy to see that

$$K_0(t)-\Phi^P_t(K)=\int_0^t\Phi_{t-\tau}(R(\tau))d\tau+\Phi_t(R_0),$$

which immediately implies that $K_0(t)-\Phi^P_t(K)\in\Psi^{m-1,-\infty}(M,\mathcal{F},E)$. □

7. Preliminaries on transverse Dirac operators

Let M be a compact manifold equipped with a Riemannian foliation $\mathcal{F}$ of even codimension q and $\mathcal{E}$ a Hermitian vector bundle over M equipped with a leafwise flat unitary connection $\nabla^{\mathcal{E}}$. Suppose that g_M is a bundle-like metric on M.

As above, let $T^H_xM=T_x\mathcal{F}^\perp$. Let P_H (resp. P_F) denote the orthogonal projection operator of $TM=T^HM\oplus T\mathcal{F}$ on T^HM (resp. $T\mathcal{F}$). There is the canonical flat connection $\overset{\circ}{\nabla}$ in T^HM, defined along the leaves of $\mathcal{F}$ (the Bott connection) given by

$$\overset{\circ}{\nabla}_XN=P_H[X,N],\quad X\in C^\infty(M,T\mathcal{F}),N\in C^\infty(M,T^HM).$$

Denote by ∇^L the Levi-Civita connection defined by g_M. The following formulas define a connection ∇ in T^HM (called the transverse Levi-Civita connection):

$$\begin{aligned}&\nabla_XN=P_H[X,N],\quad X\in C^\infty(M,T\mathcal{F}),N\in C^\infty(M,T^HM)\\ &\nabla_XN=P_H\nabla^L_XN,\quad X\in C^\infty(M,T^HM),N\in C^\infty(M,T^HM).\end{aligned}\tag{11}$$

It turns out that ∇ depends only on the transverse part of the metric g_M and preserves the inner product of T^HM. This connection will be called the transverse Levi-Civita connection.

Denote by $\mathcal{R}$ the integrability tensor (or curvature) of T^HM. It is the 2-form on T^HM with values in $T\mathcal{F}$ given by

$$\mathcal{R}_x(f_1,f_2)=-P_F[\tilde{f}_1,\tilde{f}_2](x),\quad f_1,f_2\in T^H_xM,$$

where, for any $f \in T^H_x M$, $\tilde{f} \in C^\infty(M, T^H M)$ denotes any infinitesimal transformation of $\mathcal{F}$, which coincides with f at x.

Since the Levi-Civita connection ∇^L has no torsion, for any $f_1, f_2 \in C^\infty(M, T^H M)$, we have

$$\nabla_{f_1} f_2 - \nabla_{f_2} f_1 = P_H([f_1, f_2]) = [f_1, f_2] + \mathcal{R}(f_1, f_2). \tag{12}$$

Let $\omega_\mathcal{F}$ denote the leafwise Riemannian volume form of $\mathcal{F}$. Let $f \in T^H_x M$ and let $\tilde{f} \in C^\infty(M, T^H M)$ denote any infinitesimal transformation of $\mathcal{F}$, which coincides with f at x. The local flow generated by $\tilde{f}$ preserves the foliation and gives rise to a well-defined action on $\Lambda^p T^*\mathcal{F}$. The mean curvature vector field $\tau \in C^\infty(M, T^H M)$ of $\mathcal{F}$ is defined by the identity

$$L_{\tilde{f}} \omega_\mathcal{F} = g_M(\tau, \tilde{f}) \omega_\mathcal{F}.$$

If $e_1, e_2, \dots, e_p$ is a local orthonormal frame in $T\mathcal{F}$, then

$$\tau = \sum_{i=1}^p P_H(\nabla^L_{e_i} e_i).$$

Assume that $\mathcal{F}$ is transversely oriented and the normal bundle Q is spin. Thus the $SO(q)$ bundle $O(Q)$ of oriented orthonormal frames in Q can be lifted to a $\mathrm{Spin}(q)$ bundle $O'(Q)$ so that the projection $O'(Q) \to O(Q)$ induces the covering projection $\mathrm{Spin}(q) \to SO(q)$ on each fiber.

Let $F(Q)$, $F_+(Q)$, $F_-(Q)$ be the bundles of spinors

$$F(Q) = O'(Q) \times_{\mathrm{Spin}(q)} S, \qquad F_\pm(Q) = O'(Q) \times_{\mathrm{Spin}(q)} S_\pm.$$

Denote by $\mathrm{Cl}(Q_x)$ the Clifford algebra of Q_x, $x \in M$. Recall that, relative to an orthonormal basis $\{f_1, f_2, \dots, f_q\}$ of Q_x, $\mathrm{Cl}(Q_x)$ is the complex algebra generated by 1 and $f_1, f_2, \dots, f_q$ with relations

$$f_\alpha f_\beta + f_\beta f_\alpha = -2\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \dots, q.$$

Since $\dim Q = q$ is even $\mathrm{End}\, F(Q)$ is as a bundle of algebras over M isomorphic to the Clifford bundle $\mathrm{Cl}(Q)$. The action of an element $a \in \mathrm{Cl}(Q)$ on $F(Q)$ will be denoted by $c(a)$.

The transverse Levi-Civita connection ∇ lifts to a connection $\nabla^{F(Q)}$ on the holonomy equivariant vector bundle $F(Q)$, whose restriction to $T\mathcal{F}$ coincides with the Bott connection. It can be easily seen that $\nabla^{F(Q)}$ is a Clifford connection, that is, for any $f \in T^H M$ and $X \in T^H M$, we have

$$[\nabla^{F(Q)}_f, c(X)] = c(\nabla_f X).$$

Let

$$\nabla^{F(Q)\otimes\mathcal{E}} = \nabla^{F(Q)} \otimes 1 + 1 \otimes \nabla^\mathcal{E}$$

be the corresponding connection on $F(Q) \otimes \mathcal{E}$.

We will identify the bundle Q and Q^* by means of the metric g_M and define the operator $D'_\mathcal{E}$ acting on the sections of $F(Q) \otimes \mathcal{E}$ as the composition

$$\begin{aligned} C^\infty(M, F(Q)\otimes\mathcal{E}) &\overset{\nabla^{F(Q)\otimes\mathcal{E}}}{\longrightarrow} C^\infty(M, Q^* \otimes F(Q)\otimes\mathcal{E}) \\ &= \quad C^\infty(M, Q \otimes F(Q)\otimes\mathcal{E}) \overset{c\otimes 1}{\longrightarrow} C^\infty(M, F(Q)\otimes\mathcal{E}). \end{aligned}$$

This operator is odd with respect to the $\mathbb{Z}_2$-grading $F(Q) \otimes \mathcal{E} = (F_+(Q) \otimes \mathcal{E}) \oplus (F_-(Q) \otimes \mathcal{E})$. If $f_1, \dots, f_q$ is a local orthonormal frame for $T^H M$, then

$$D'_\mathcal{E} = \sum_{\alpha=1}^q (c(f_\alpha) \otimes 1) \nabla^{F(Q)\otimes\mathcal{E}}_{f_\alpha}.$$

Denote by $(\cdot,\cdot)_x$ the inner product in the fiber $(F(Q)\otimes\mathcal{E})_x$ over $x\in M$. Then the inner product in $L^2(M,F(Q)\otimes\mathcal{E})$ is given by the formula

$$(s_1,s_2)=\int_M (s_1(x),s_2(x))_x\omega_M,\quad s_1,s_2\in L^2(M,F(Q)\otimes\mathcal{E}),$$

where $\omega_M=\sqrt{\det g}\,\mathrm{d}x$ denotes the Riemannian volume form on M. In the following lemma, we compute the formal adjoint $(D'_{\mathcal{E}})^*$ of $D'_{\mathcal{E}}$ (see also [11] and references therein).

Lemma 7.1. *We have*

$$(D'_{\mathcal{E}})^*=D'_{\mathcal{E}}-c(\tau).$$

Proof. For any $s_1,s_2\in C^\infty(M,F(Q)\otimes\mathcal{E})$, we have

$$\begin{aligned}(D'_{\mathcal{E}}s_1,s_2)&=\sum_{\alpha=1}^{q}\left(-\int_M f_\alpha[(s_1,(c(f_\alpha)\otimes 1)s_2)_x]\omega_M+(s_1,\nabla^{F(Q)\otimes\mathcal{E}}_{f_\alpha}(c(f_\alpha)\otimes 1)s_2)\right)\\&=\sum_{\alpha=1}^{q}\left(-\int_M f_\alpha[(s_1,(c(f_\alpha)\otimes 1)s_2)_x]\omega_M+(s_1,(c(\nabla_{f_\alpha}f_\alpha)\otimes 1)s_2)\right)+(s_1,D'_{\mathcal{E}}s_2).\end{aligned}$$

Recall that, by the divergence theorem, for any vector field X on M and $a\in C^\infty(M)$, we have

$$\int_M X(a)(x)\omega_M=-\int_M \operatorname{div}(X)\cdot a(x)\omega_M.$$

Let $e_1,e_2,\ldots,e_p$ be a local orthonormal frame in $T\mathcal{F}$. Then the divergence $\operatorname{div}(X)$ of X is given by the formula

$$\operatorname{div}(X)=\sum_{k=1}^{p}g_M(e_k,\nabla_{e_k}X)+\sum_{\beta=1}^{q}g_M(f_\beta,\nabla_{f_\beta}X).\tag{13}$$

In particular, it is easy to see that

$$\operatorname{div}(f_\alpha)=-g_M\left(\tau+\sum_{\beta=1}^{q}\nabla_{f_\beta}f_\beta,f_\alpha\right).$$

Using the divergence theorem, we easily get

$$\sum_{\alpha=1}^{q}\left(-\int_M f_\alpha[(s_1,(c(f_\alpha)\otimes 1)s_2)_x]\omega_M+(s_1,(c(\nabla_{f_\alpha}f_\alpha)\otimes 1)s_2)\right)=-(s_1,(c(\tau)\otimes 1)s_2),$$

which completes the proof. □

By this lemma, the operator

$$D_{\mathcal{E}}=D'_{\mathcal{E}}-\frac{1}{2}c(\tau)=\sum_{\alpha=1}^{q}(c(f_\alpha)\otimes 1)\left(\nabla^{F(Q)\otimes\mathcal{E}}_{f_\alpha}-\frac{1}{2}g_M(\tau,f_\alpha)\right)$$

is self-adjoint. This operator will be called the transverse Dirac operator. It was introduced in [11] (see also [12,13] and references therein).

We will use the Riemannian volume form ω_M to identify the half-densities bundle with the trivial one. So the action of $D_{\mathcal{E}}$ on half-densities is defined by

$$D_{\mathcal{E}}(u|\omega_M|^{1/2})=(D_{\mathcal{E}}u)|\omega_M|^{1/2},\quad u\in C^\infty(M,F(Q)\otimes\mathcal{E}).$$

8. The transverse signature operator

In this section, we will discuss a particular example of a transverse Dirac operator given by the transverse signature operator.

As above, let $(M, \mathcal{F})$ be a compact Riemannian foliated manifold equipped with a bundle-like metric g_M.

Lemma 8.1. *Let $f_1, f_2, \ldots, f_q$ be a local orthonormal basis of $T^H M$ and $f_1^*, f_2^*, \ldots, f_q^*$ be the dual basis of $T^H M^*$. Then on $C^\infty(M, \Lambda T^H M^*)$ we have*

$$\mathrm{d}_H = \sum_{\alpha=1}^{q} \varepsilon_{f_\alpha^*} \nabla_{f_\alpha},$$

$$\mathrm{d}_H^* = -\sum_{\alpha=1}^{q} i_{f_\alpha} \nabla_{f_\alpha} + i_\tau.$$

Proof. Denote $\mathrm{d}'_H = \sum_{\alpha=1}^{q} \varepsilon_{f_\alpha^*} \nabla_{f_\alpha}$. Then the operators d_H and d'_H satisfy the Leibniz rule and, clearly, coincide on functions. It remains to show that they agree on the space $C^\infty(M, T^H M^*)$ of transverse 1-forms. Using an explicit formula for d_H and (12), for any $\omega \in C^\infty(M, T^H M^*)$ and for any $U, V \in C^\infty(M, T^H M)$, we get

$$\begin{aligned}\mathrm{d}_H\omega(U, V) &= U[\omega(V)] - V[\omega(U)] - \omega(P_H[U, V]) \\ &= \nabla_U\omega(V) + \omega(\nabla_U V) - \nabla_V\omega(U) - \omega(\nabla_V U) - \omega(P_H[U, V]) \\ &= \nabla_U\omega(V) - \nabla_V\omega(U).\end{aligned}$$

Now, since $U = \sum_\alpha \langle f_\alpha^*, U\rangle f_\alpha$ and $V = \sum_\alpha \langle f_\alpha^*, V\rangle f_\alpha$, we obtain

$$\nabla_U\omega(V) - \nabla_V\omega(U) = \sum_\alpha (\langle f_\alpha^*, U\rangle \nabla_{f_\alpha}\omega(V) - \langle f_\alpha^*, V\rangle \nabla_{f_\alpha}\omega(U)) = \mathrm{d}'_H\omega(U, V),$$

which proves the first equality.

The second equality can be easily derived from the first one, if we take the adjoints and use the divergence theorem. □

To represent the transverse signature operator $D_H = \mathrm{d}_H + \mathrm{d}_H^*$ as a transverse Dirac operator, we take $\mathcal{E} = F(Q)^*$. By Lemma 8.1, we have the following formula for the corresponding transverse Dirac operator $D_{F(Q)^*}$:

$$\begin{aligned}D_{F(Q)^*} &= \sum_{\alpha=1}^{q} (\varepsilon_{f_\alpha^*} - i_{f_\alpha}) \nabla_{f_\alpha} - \frac{1}{2}(\varepsilon_{\tau^*} - i_\tau) \\ &= \mathrm{d}_H + \mathrm{d}_H^* - \frac{1}{2}(\varepsilon_{\tau^*} + i_\tau).\end{aligned}$$

So we see that the transverse signature operator D_H coincides with the transverse Dirac operator $D_{F(Q)^*}$ if and only if $\tau = 0$, that is, all the leaves are minimal submanifolds.

Example 8.2. Consider a foliation $\mathcal{F}$ on a compact manifold M given by the fibers of a principal K-bundle $f : M \to B$ with connection, where K is a compact group. Fix a Riemannian metric on B and a bi-invariant metric on K, and consider the corresponding Kaluza–Klein metric on M.

For any irreducible unitary representation ρ of K in a vector space W_ρ, consider the associated Hermitian vector bundle $E_\rho = M \times_\rho W_\rho$ over B. It is well known that there is a natural identification of the space $C^\infty(B, E_\rho)$ with the space F_ρ of smooth functions $f : M \to W_\rho$ satisfying $f(x \cdot k) = \rho(k)^{-1} f(x)$ for any $x \in M$ and $k \in K$. We denote by $C^\infty(M)_\rho$ the isotypical component of ρ in $C^\infty(M)$. So we have

$$C^\infty(M) = \bigoplus_{\rho \in \hat{K}} C^\infty(M)_\rho.$$

The following lemma is a generalization of the usual Peter–Weyl theorem to bundles (see, for instance, [15, Lemma 5.3]).

Lemma 8.3. *The mapping*

$$J_\rho : C^\infty(B, E_\rho) \otimes W_\rho^* \to C^\infty(M), \quad f \otimes \eta \mapsto f_\eta,$$

where

$$f_\eta(x) = \sqrt{\frac{\dim W_\rho}{\operatorname{vol} K}}\, \eta(f(x)), \quad x \in M,$$

is a unitary isomorphism onto $C^\infty(M)_\rho$, which is K-equivariant with respect to the representation $1 \otimes \rho^*$ on $C^\infty(B, E_\rho) \otimes W_\rho^*$.

Next, the transverse de Rham differential d_H commutes with the natural action of K on $C^\infty(M, \Lambda^* T^H M^*)$. Therefore, d_H maps $C^\infty(M, \Lambda^* T^H M^*)_\rho$ to $C^\infty(M, \Lambda^* T^H M^*)_\rho$. Let $\nabla^{E_\rho} : C^\infty(B, \Lambda^* T^* B \otimes E_\rho) \to C^\infty(B, \Lambda^* T^* B \otimes E_\rho)$ be the exterior covariant derivative associated with the connection. By definition (see, for instance [20]), under the isomorphism J_ρ, the restriction of d_H to $C^\infty(M, \Lambda^* T^H M^*)_\rho$ corresponds to the operator $\nabla^{E_\rho} \otimes I_{W_\rho^*}$ on $C^\infty(B, \Lambda^* T^* B \otimes E_\rho) \otimes W_\rho^*$. Since the isomorphism J_ρ is unitary, a similar statement holds for d_H^*.

Thus, we have the commutative diagram

$$\begin{array}{ccc} C^\infty(B, \Lambda^* T^* B \otimes E_\rho) \otimes W_\rho^* & \xrightarrow{D^{E_\rho} \otimes I_{W_\rho^*}} & C^\infty(B, \Lambda^* T^* B \otimes E_\rho) \otimes W_\rho^* \\ \downarrow & & \downarrow \\ C^\infty(M, \Lambda^* T^H M^*)_\rho & \xrightarrow{D_H} & C^\infty(M, \Lambda^* T^H M^*)_\rho \end{array}$$

where $D^{E_\rho} = \nabla^{E_\rho} + (\nabla^{E_\rho})^*$ is the twisted signature operator on B with coefficients in the vector bundle E_ρ. It shows that, in this case, the transverse signature operator $D_H = d_H + d_H^*$ decomposes into a direct sum of twisted signature operators on the base B with coefficients in vector bundles associated with irreducible representations of K.

9. The subprincipal symbol of a transverse Dirac operator

In this section we compute the transverse bicharacteristic flow of transverse Dirac operators. For this, we will use the following fact (see, for instance, [6, Proposition 4.3.1]).

Theorem 9.1. *Let $P \in \Psi^m(X)$ be a properly supported pseudodifferential operator on a smooth manifold X. For any $a \in C^\infty(X, |TX|^{1/2})$ and for any real-valued function $\phi \in C^\infty(X)$ we have*

$$e^{-is\phi(x)} P(e^{is\phi} a)(x) = s^m p_m(x, d\phi(x)) \cdot a(x) + s^{m-1} \left(p_{\mathrm{sub}}(x, d\phi(x)) \cdot a(x) + \frac{1}{i} (\mathcal{L}_v a)(x) \right) + O(s^{m-2}),$$
$$s \to \infty,$$

where v is a vector field on X:

$$v(x) = \sum_j \frac{\partial p_m}{\partial \xi_j}(x, d\phi(x)) \frac{\partial}{\partial x_j} = \pi_*(X_p(x, d\phi(x))),$$

X_p is the Hamiltonian vector field of p_m on T^*X, $\pi_*(X_p(x,\xi)) \in T_x X$ is the image of $X_p(x,\xi) \in T_{(x,\xi)}(T^*X)$ under the projection $\pi : T^*X \to X$.

Here $\mathcal{L}_v$ denotes the Lie derivative along v, acting on half-densities: for any $f \in C^\infty(X)$, we have

$$\mathcal{L}_v(f|\omega_X|^{1/2}) = v(f)|\omega_X|^{1/2} + \frac{1}{2} \operatorname{div} v \cdot f|\omega_X|^{1/2}.$$

This theorem remains true for operators acting on sections of a vector bundle E over X locally, that is, if we fix a trivialization of E over some open subset of X.

Let M be a compact manifold equipped with a Riemannian foliation $\mathcal{F}$ of even codimension q, $\mathcal{E}$ a Hermitian vector bundle over M equipped with a leafwise flat unitary connection $\nabla^{\mathcal{E}}$, g_M a bundle-like metric on M and $D_{\mathcal{E}}$ the associated transverse Dirac operator.

Theorem 9.2. *The principal symbol of* $D_{\mathcal{E}}^2$ *is given by*

$$a_2(x,\xi) = g^M(P^H(x,\xi), P^H(x,\xi)), \quad (x,\xi) \in T^*M, \tag{14}$$

where g^M *is the induced metric on* T^*M, $P^H : T^*M \to T^HM^*$ *is the orthogonal projection.*

Proof. Let $f_1, \ldots, f_q$ be a local orthonormal basis of T^HM, which consists of infinitesimal transformations of $\mathcal{F}$. For any $a \in C^\infty(M, F(Q) \otimes \mathcal{E})$ and for any real-valued function $\phi \in C^\infty(M)$ we have

$$\mathrm{e}^{-\mathrm{i}s\phi(x)} D_{\mathcal{E}}^2(\mathrm{e}^{\mathrm{i}s\phi}a)(x) = \left(\sum_{\alpha=1}^{q}(c(f_\alpha)\otimes 1)\left(\nabla_{f_\alpha}^{F(Q)\otimes\mathcal{E}} - \frac{1}{2}g_M(\tau, f_\alpha) + \mathrm{i}sf_\alpha(\phi)\right)\right)^2. \tag{15}$$

The terms of order s^2 in (15) are

$$s^2\sum_{\alpha=1}^{q}(f_\alpha(\phi))^2 = s^2\sum_{\alpha=1}^{q}\langle \mathrm{d}\phi, f_\alpha\rangle^2.$$

Therefore, the principal symbol of $D_{\mathcal{E}}^2$ is

$$a_2(x,\xi) = \sum_{\alpha=1}^{q}\langle \xi, f_\alpha(x)\rangle^2, \quad (x,\xi) \in T^*M,$$

which completes the proof. □

It is easy to see that the 1-jets of the functions $\sqrt{a_2}$, where a_2 is the principal symbol of $D_{\mathcal{E}}^2$, and

$$p(x,\xi) = \sqrt{g^M((x,\xi),(x,\xi))}, \quad (x,\xi) \in T^*M,$$

coincide on $N^*\mathcal{F}$. So we can further work with the elliptic symbol p.

The Hermitian connection $\nabla^{F(Q)\otimes\mathcal{E}}$ determines uniquely a Hermitian partial connection $\tilde{\nabla}_{X_p}$ along the Hamiltonian vector field X_p on $\pi^*(F(Q)\otimes\mathcal{E})$, which satisfies

$$\left(\tilde{\nabla}_{X_p}\pi^*s\right)(\nu) = \nabla_{\pi_*(X_p(\nu))}^{F(Q)\otimes\mathcal{E}}s(\pi(\nu)), \quad s \in C^\infty(M, F(Q)\otimes\mathcal{E}),$$

where $\pi^*s \in C^\infty(\tilde{N}^*\mathcal{F}, \pi^*(F(Q)\otimes\mathcal{E}))$ is the pull back of a section $s \in C^\infty(M, F(Q)\otimes\mathcal{E})$ under the projection $\pi : \tilde{N}^*\mathcal{F} \to M$.

If we fix a local orthonormal basis $f_1, \ldots, f_q$ of T^HM, which consists of infinitesimal transformations of $\mathcal{F}$, and a local trivialization of $F(Q)\otimes\mathcal{E}$ and write $\nabla_{f_\alpha}^{F(Q)\otimes\mathcal{E}} = f_\alpha + B(f_\alpha)$ with some matrix-valued 1-form B, then, for $s \in C^\infty(\tilde{N}^*\mathcal{F}, \pi^*(F(Q)\otimes\mathcal{E}))$, we have

$$(\tilde{\nabla}_{X_p}s)(\nu) = X_ps(\nu) + \|\nu\|^{-1}\sum_{\alpha=1}^{q}\langle \nu, f_\alpha\rangle B(f_\alpha)s(\nu), \quad \nu \in N^*\mathcal{F}.$$

The geometric meaning of this partial connection is as follows. Recall that X_p generates the geodesic flow f_t on $\tilde{N}^*\mathcal{F}$. For any $\nu \in \tilde{N}^*\mathcal{F}$, the projection of the orbit $\mathcal{O}_\nu = \{f_t(\nu), t \in \mathbb{R}\}$ to M is the geodesic γ_ν, passing through $x = \pi(\nu)$. Then the parallel transport of $v \in \pi^*(F(Q)\otimes\mathcal{E})_\nu$ along $\mathcal{O}_\nu$ with respect to the connection $\tilde{\nabla}_{X_p}$ coincides with the parallel transport of v considered as an element of $(F(Q)\otimes\mathcal{E})_x$ along the geodesic γ_ν with respect to the connection $\nabla^{F(Q)\otimes\mathcal{E}}$.

Theorem 9.3. *The subprincipal symbol of* $\langle D_{\mathcal{E}}\rangle$ *considered as a partial connection* $\nabla_{\mathrm{sub}}(\langle D_{\mathcal{E}}\rangle)$ *on* $\pi^*(F(Q)\otimes\mathcal{E})$ *coincides with* $\tilde{\nabla}_{X_p}$.

Proof. The terms of order s in (15) are

$$\begin{aligned}
&\mathrm{i}\left[\left(\sum_{\alpha=1}^{q}(c(f_\alpha)\otimes 1)f_\alpha(\phi)\right)\left(\sum_{\beta=1}^{q}(c(f_\beta)\otimes 1)\left(\nabla^{F(Q)\otimes\mathcal{E}}_{f_\beta}-\frac{1}{2}g_M(\tau,f_\beta)\right)\right)\right.\\
&\quad\left.+\left(\sum_{\beta=1}^{q}(c(f_\beta)\otimes 1)\left(\nabla^{F(Q)\otimes\mathcal{E}}_{f_\beta}-\frac{1}{2}g_M(\tau,f_\beta)\right)\right)\left(\sum_{\alpha=1}^{q}(c(f_\alpha)\otimes 1)f_\alpha(\phi)\right)\right]\\
&=\mathrm{i}\sum_{\alpha,\beta}((c(f_\alpha)c(f_\beta)+c(f_\beta)c(f_\alpha))\otimes 1)f_\alpha(\phi)\left(\nabla^{F(Q)\otimes\mathcal{E}}_{f_\beta}-\frac{1}{2}g_M(\tau,f_\beta)\right)\\
&\quad+\mathrm{i}\sum_{\alpha,\beta}(c(f_\beta)c(\nabla_{f_\beta}f_\alpha)\otimes 1)f_\alpha(\phi)+\mathrm{i}\sum_{\alpha,\beta}(c(f_\beta)c(f_\alpha)\otimes 1)f_\beta f_\alpha(\phi)\\
&=I_1+I_2+I_3.
\end{aligned}$$

For the first term, we easily get

$$I_1=-2\mathrm{i}\sum_{\alpha}f_\alpha(\phi)\nabla^{F(Q)\otimes\mathcal{E}}_{f_\alpha}-\mathrm{i}\tau(\phi).$$

Let $\nabla_{f_\alpha}f_\beta=\sum_\gamma a^\gamma_{\alpha\beta}f_\gamma$. Since ∇ is compatible with the metric, we have $a^\gamma_{\alpha\beta}=-a^\beta_{\alpha\gamma}$. Thus we get

$$\begin{aligned}
I_2&=\frac{\mathrm{i}}{2}\sum_{\alpha,\beta}[(c(f_\alpha)c(\nabla_{f_\alpha}f_\beta)\otimes 1)f_\beta(\phi)+(c(f_\beta)c(\nabla_{f_\beta}f_\alpha)\otimes 1)f_\alpha(\phi)]\\
&=\frac{\mathrm{i}}{2}\sum_{\alpha,\beta,\gamma}[a^\gamma_{\alpha\beta}(c(f_\alpha)c(f_\gamma)\otimes 1)f_\beta(\phi)+a^\gamma_{\alpha\beta}(c(f_\beta)c(f_\gamma)\otimes 1)f_\alpha(\phi)]\\
&=-\frac{1}{2}\sum_{\alpha,\beta,\gamma}[a^\beta_{\alpha\gamma}(c(f_\alpha)c(f_\gamma)\otimes 1)f_\beta(\phi)+a^\alpha_{\beta\gamma}(c(f_\beta)c(f_\gamma)\otimes 1)f_\alpha(\phi)]\\
&=-\frac{\mathrm{i}}{2}\sum_{\alpha,\gamma}[(c(f_\alpha)c(f_\gamma)\otimes 1)\nabla_{f_\alpha}f_\gamma(\phi)+\sum_{\beta,\gamma}(c(f_\beta)c(f_\gamma)\otimes 1)\nabla_{f_\beta}f_\gamma(\phi)]\\
&=-\mathrm{i}\sum_{\alpha,\beta}(c(f_\alpha)c(f_\beta)\otimes 1)\nabla_{f_\alpha}f_\beta(\phi).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
I_3&=\frac{\mathrm{i}}{2}\sum_{\alpha,\beta}((c(f_\beta)c(f_\alpha)\otimes 1)f_\beta f_\alpha(\phi)+(c(f_\alpha)c(f_\beta)\otimes 1)f_\alpha f_\beta(\phi))\\
&=\frac{\mathrm{i}}{2}\sum_{\alpha,\beta}((c(f_\beta)c(f_\alpha)+c(f_\alpha)c(f_\beta))\otimes 1)f_\alpha f_\beta(\phi)+\frac{\mathrm{i}}{2}\sum_{\alpha,\beta}(c(f_\beta)c(f_\alpha)\otimes 1)[f_\beta,f_\alpha](\phi)\\
&=-\mathrm{i}\sum_{\alpha}f_\alpha^2(\phi)+\frac{\mathrm{i}}{2}\sum_{\alpha,\beta}(c(f_\beta)c(f_\alpha)\otimes 1)(\nabla_{f_\beta}f_\alpha-\nabla_{f_\alpha}f_\beta-\mathcal{R}(f_\beta,f_\alpha))(\phi),
\end{aligned}$$

where we used the equality (12).

From the last three identities, the terms of order s are

$$-2\mathrm{i}\sum_{\alpha}f_\alpha(\phi)\nabla^{F(Q)\otimes\mathcal{E}}_{f_\alpha}-\mathrm{i}\tau(\phi)-\mathrm{i}\sum_{\alpha}f_\alpha^2(\phi)+\mathrm{i}\sum_{\alpha}\nabla_{f_\alpha}f_\alpha(\phi)-\frac{\mathrm{i}}{2}\sum_{\alpha,\beta}(c(f_\beta)c(f_\alpha)\otimes 1)\mathcal{R}(f_\beta,f_\alpha)(\phi).$$

By Theorem 9.1, we have

$$p_{\text{sub}}(x, \mathrm{d}\phi(x)) \cdot a(x)|\omega_M|^{1/2} + \frac{1}{\mathrm{i}}\mathcal{L}_v(a|\omega_M|^{1/2})(x) = \left(-\mathrm{i}\sum_{\alpha=1}^{q} f_\alpha^2(\phi)a - 2\mathrm{i}\sum_{\alpha=1}^{q} f_\alpha(\phi)\nabla_{f_\alpha}^{F(Q)\otimes\mathcal{E}}a - \mathrm{i}\tau(\phi)a \right.$$
$$\left. + \mathrm{i}\sum_{\alpha=1}^{q}\nabla_{f_\alpha} f_\alpha(\phi)a - \frac{1}{2}\mathrm{i}\sum_{\alpha=1}^{q}\sum_{\beta=1}^{q} c(f_\alpha)c(f_\beta)\mathcal{R}(f_\alpha, f_\beta)(\phi)a\right)|\omega_M|^{1/2}.$$

Now compute the vector field v:

$$\pi_*(X_p(x,\xi)) = 2\sum_{\alpha=1}^{q}\langle\xi, f_\alpha\rangle f_\alpha, \quad \xi \in T^*M$$

and

$$v = 2\sum_{\alpha=1}^{q}\langle \mathrm{d}\phi(x), f_\alpha\rangle f_\alpha = 2\sum_{\alpha=1}^{q} f_\alpha(\phi) f_\alpha.$$

Therefore, we have

$$\mathcal{L}_v(a|\omega_M|^{1/2}) = 2\sum_{\alpha=1}^{q} f_\alpha(\phi) f_\alpha(a)|\omega_M|^{1/2} + \sum_{\alpha=1}^{q}\operatorname{div}(f_\alpha(\phi) f_\alpha)\cdot a|\omega_M|^{1/2}.$$

Let $e_1, e_2, \ldots, e_p$ be a local orthonormal frame in $T\mathcal{F}$. Using (13), we easily compute

$$\sum_{\alpha=1}^{q}\operatorname{div}(f_\alpha(\phi) f_\alpha) = \sum_{\alpha=1}^{q} f_\alpha^2(\phi) - \tau(\phi) - \sum_{\beta=1}^{q}\nabla_{f_\beta} f_\beta(\phi).$$

Finally, if we write $\nabla_{f_\alpha}^{F(Q)\otimes\mathcal{E}} = f_\alpha + B(f_\alpha)$ with some matrix-valued 1-form B, we get

$$p_{\text{sub}}(x, \mathrm{d}\phi(x)) = -2\mathrm{i}\sum_{\alpha=1}^{q} f_\alpha(\phi)B(f_\alpha) - \frac{\mathrm{i}}{2}\sum_{\alpha=1}^{q}\sum_{\beta=1}^{q} c(f_\alpha)c(f_\beta)\mathcal{R}(f_\alpha, f_\beta)(\phi),$$

or, equivalently,

$$p_{\text{sub}}(x,\xi) = -2\mathrm{i}\sum_{\alpha=1}^{q}\langle\xi, f_\alpha\rangle B(f_\alpha) - \frac{\mathrm{i}}{2}\sum_{\alpha=1}^{q}\sum_{\beta=1}^{q} c(f_\alpha)c(f_\beta)\langle\xi, \mathcal{R}(f_\alpha, f_\beta)\rangle.$$

The transverse subprincipal symbol of $D_{\mathcal{E}}^2$ is

$$\sigma_{\text{sub}}(D_{\mathcal{E}}^2)(\nu) = -2\mathrm{i}\sum_{\alpha=1}^{q}\langle\nu, f_\alpha\rangle B(f_\alpha), \quad \nu \in N^*\mathcal{F}.$$

Since the principal symbol of $D_{\mathcal{E}}^2$ is scalar, the formula proved in [7]

$$\sigma_{\text{sub}}(\langle D_{\mathcal{E}}\rangle) = \frac{1}{2}\sigma(D_{\mathcal{E}}^2)^{-\frac{1}{2}}\sigma_{\text{sub}}(D_{\mathcal{E}}^2),$$

continues to hold and the transverse subprincipal symbol of $\langle D_{\mathcal{E}}\rangle$ is

$$\sigma_{\text{sub}}(\langle D_{\mathcal{E}}\rangle)(\nu) = -\mathrm{i}\|\nu\|^{-1}\sum_{\alpha=1}^{q}\langle\nu, f_\alpha\rangle B(f_\alpha), \quad \nu \in N^*\mathcal{F}.$$

Thus, the subprincipal symbol of $\langle D_{\mathcal{E}}\rangle$ is a partial connection $\nabla_{\text{sub}}(\langle D_{\mathcal{E}}\rangle)$ on π^*E along the Hamiltonian vector field X_p given by

$$\nabla_{\text{sub}}(\langle D_{\mathcal{E}}\rangle) = X_p(\nu) + \|\nu\|^{-1}\sum_{\alpha=1}^{q}\langle\nu, f_\alpha\rangle B(f_\alpha) = \tilde{\nabla}_{X_p(\nu)}, \quad \nu \in N^*\mathcal{F},$$

as desired. □

10. The noncommutative geodesic flow

As stated in [21] (see also [23]), any operator D satisfying the assumptions of Section 6 defines a spectral triple in the sense of Connes' noncommutative geometry [2,3]. In this setting, Theorem 6.1 has a natural interpretation in terms of the corresponding noncommutative geodesic flow.

More precisely, consider spectral triples $(\mathcal{A}, \mathcal{H}, D)$ associated with a compact foliated Riemannian manifold $(M, \mathcal{F})$ (see [23] for more details):

(1) The involutive algebra $\mathcal{A}$ is the algebra $C^\infty_c(G, |T\mathcal{G}|^{1/2})$.
(2) The Hilbert space $\mathcal{H}$ is the space $L^2(M, E)$ of L^2-sections of a holonomy equivariant Hermitian vector bundle E, on which an element k of the algebra $\mathcal{A}$ is represented via the $*$-representation R_E.
(3) The operator D is a first-order self-adjoint transversally elliptic operator with the holonomy invariant transversal principal symbol such that the operator D^2 has the scalar principal symbol.

Let $S^*\mathcal{A}$ denote the unitary cotangent bundle and γ_t the noncommutative geodesic flow associated with $(\mathcal{A}, \mathcal{H}, D)$ (see [23] for definitions in the non-unital case). Thus, $S^*\mathcal{A}$ is a C^*-algebra and γ_t is a one-parameter group of its automorphisms.

The transversal bicharacteristic flow $\mathrm{Ad}(\alpha_t)^*$ of the operator $\langle D\rangle$ extends by continuity to a strongly continuous one-parameter group of automorphisms of the algebra $\bar{S}^0(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$, the uniform closure of $S^0(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$ (see [23]).

Theorem 10.1. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple associated with a compact foliated Riemannian manifold $(M, \mathcal{F})$ as above. There exists a surjective homomorphism of involutive algebras*

$$P : S^*\mathcal{A} \to \bar{S}^0(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2})$$

such that the following diagram commutes:

$$\begin{array}{ccc} S^*\mathcal{A} & \xrightarrow{\gamma_t} & S^*\mathcal{A} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P} \\ \bar{S}^0(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2}) & \xrightarrow{\mathrm{Ad}(\alpha_t)^*} & \bar{S}^0(G_{\mathcal{F}_N}, \mathcal{L}(\pi^*E) \otimes |T\mathcal{G}_N|^{1/2}) \end{array} \qquad (16)$$

Here the map P is induced by the principal symbol map $\bar{\sigma}$, and the theorem is a simple consequence of the results of [23] and Theorem 6.1.

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References

[1] J. Bolte, R.A. Glaser, Semiclassical Egorov theorem and quantum ergodicity for matrix valued operators, Comm. Math. Phys. 247 (2004) 391–419.
[2] A. Connes, Geometry from the spectral point of view, Lett. Math. Phys. 34 (1995) 203–238.
[3] A. Connes, H. Moscovici, The local index formula in noncommutative geometry, Geom. Funct. Anal. 5 (1995) 174–243.
[4] N. Dencker, On the propagation of polarization sets for systems of real principal type, J. Funct. Anal. 46 (1982) 351–372.
[5] R.G. Douglas, J.F. Glazebrook, F.W. Kamber, G. Yu, Index formulas for geometric Dirac operators in Riemannian foliations, K-Theory 9 (1995) 407–441.
[6] J.J. Duistermaat, Fourier Integral Operators, Birkhäuser, Boston, 1996.
[7] J.J. Duistermaat, V. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, Invent. Math. 29 (1975) 39–79.
[8] Ju.V. Egorov, The canonical transformations of pseudodifferential operators, Uspekhi Mat. Nauk 24 (5) (1969) 235–236.
[9] C. Emmrich, A. Weinstein, Geometry of the transport equation in multicomponent WKB approximations, Comm. Math. Phys. 176 (1996) 701–711.
[10] P. Gérard, P.A. Markowich, N.J. Mauser, F. Poupaud, Homogenization limits and Wigner transforms, Comm. Pure Appl. Math. 50 (1997) 323–379.

[11] J.F. Glazebrook, F.W. Kamber, Transversal Dirac families in Riemannian foliations, Comm. Math. Phys. 140 (1991) 217–240.
[12] J.F. Glazebrook, F.W. Kamber, On spectral flow of transversal Dirac operators and a theorem of Vafa–Witten, Ann. Global Anal. Geom. 9 (1991) 27–35.
[13] J.F. Glazebrook, F.W. Kamber, Secondary invariants and chiral anomalies of basic Dirac families, Differential Geom. Appl. 3 (1993) 285–299.
[14] V. Guillemin, S. Sternberg, Some problems in integral geometry and some related problems in microlocal analysis, Amer. J. Math. 101 (1979) 915–959.
[15] V. Guillemin, A. Uribe, Clustering theorems with twisted spectra, Math. Ann. 273 (1986) 479–506.
[16] D. Jakobson, A. Strohmaier, High energy limits of Laplace-type and Dirac-type eigenfunctions and frame flows, Comm. Math. Phys. 270 (2007) 813–833.
[17] S.D. Jung, The first eigenvalue of the transversal Dirac operator, J. Geom. Phys. 39 (2001) 253–264.
[18] S.D. Jung, Y.S. Ko, Eigenvalue estimates of the basic Dirac operator on a Riemannian foliation, Taiwanese J. Math. 10 (2006) 1139–1156.
[19] S.D. Jung, Eigenvalue estimates for the basic Dirac operator on a Riemannian foliation admitting a basic harmonic 1-form, J. Geom. Phys. 57 (2007) 1239–1246.
[20] Sh. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. I, Interscience Publ., New York, London, 1963.
[21] Yu.A. Kordyukov, Noncommutative spectral geometry of Riemannian foliations, Manuscripta Math. 94 (1997) 45–73.
[22] Yu.A. Kordyukov, The trace formula for transversally elliptic operators on Riemannian foliations, Algebra i Analiz 12 (3) (2000) 81–105. Translation in St. Petersburg Math. J. 12 (3) (2001) 407–422 (in Russian).
[23] Yu.A. Kordyukov, Egorov's theorem for transversally elliptic operators on foliated manifolds and noncommutative geodesic flow, Math. Phys. Anal. Geom. 8 (2005) 97–119.
[24] Yu.A. Kordyukov, Noncommutative geometry of foliations. E-print math.DG/0504095.
[25] P. Molino, Riemannian Foliations, Birkhäuser, Boston, 1988.
[26] B.L. Reinhart, Differential Geometry of Foliations, Springer, Berlin, 1983.
[27] M.R. Sandoval, Wave-trace asymptotics for operators of Dirac type, Comm. Partial Differential Equations 24 (1999) 1903–1944.
[28] M.R. Sandoval, The singularities of the wave trace of the basic Laplacian of a Riemannian foliation, J. Funct. Anal. 243 (2007) 1–27.